

Network synchronization: Spectral versus statistical properties

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Abstract

We consider synchronization of weighted networks, possibly with asymmetrical connections. We show that the synchronizability of the networks cannot be directly inferred from their statistical properties. Small local changes in the network structure can sensitively affect the eigenvalues relevant for synchronization, while the gross statistical network properties remain essentially unchanged. Consequently, commonly used statistical properties, including the degree distribution, degree homogeneity, average degree, average distance, degree correlation, and clustering coefficient, can fail to characterize the synchronizability of networks.

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1 Introduction

The description and classification of complex networks are often based on their statistical properties, such as the degree distribution, average degree, average distance, clustering coefficient, and degree correlations, among others [1–3]. Indeed, the starting point for the recent explosion of interest in complex networks can be traced to the observation that real networks have degree distributions that are much different from those of classical random graphs [4]. On the other hand, the dynamics of processes defined on networks are intimately related to the spectrum of an appropriate connection operator. A prototypical example is chaos synchronization [5], which crucially depends on the extremal eigenvalues of the graph Laplacian [6–8]. This raises the natural question of if and how the

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statistical properties of a network are related to its spectral properties. Many recent papers have investigated various facets of this relation. For example, some papers have reported correlations between network synchronizability and degree homogeneity [9–11], clustering coefficient [12], degree correlations [13], average degree, degree distribution, and so on [14]. In some cases the observed correlations can point in opposite directions; for instance, [9] finds that increasing the degree homogeneity improves synchronizability, whereas [14] and [13] report cases of better synchronizability for decreased homogeneity. Similarly, adding a few shortcut links to a sparse lattice is known to decrease the characteristic path length and improve synchronizability at the same time [15,16], although another study showed that better synchronization can result despite increased average distance [9]. Clearly, in view of the multitude of graph characteristics, it can be difficult to translate the numerically observed correlations into causal relations. Rigorous mathematical methods are important for investigating the relations between different network properties. The present paper provides a step in this direction. We give a mathematical argument which shows that many statistical network properties do not suffice to determine synchronizability. We present examples showing that networks with the same statistical properties can have very different synchronization characteristics. The results establish that the spectral properties of networks are not simply derivable from statistical properties, and should therefore hold their own place within the list of intrinsic network features.

2 Spectral properties and structure

Consider a network of n nodes (vertices), with links (edges) between certain pairs of nodes, which may additionally carry weights indicating the strength of the relation they represent. We use the nonnegative numbers a_{ij} to denote the weight on the link from the j th node to the i th node, where $a_{ij} = 0$ if and only if there is no link from j to i . In general $a_{ij} \neq a_{ji}$, although symmetric connections arise naturally in many common models. The degree of the vertex i is defined as $\deg(i) = \sum_{j \in V} a_{ij}$. Unweighted networks appear as a special case where each link carries a weight of 1—in this case $A = [a_{ij}]$ is the usual adjacency matrix, and the degree of a vertex is the number of its neighbors.

The Laplacian matrix is defined by $L = D - A$, where D denotes the diagonal matrix of vertex degrees. In case A is symmetric, the Laplacian is a symmetric and positive semidefinite matrix. Therefore, it has real and nonnegative eigenvalues, which we order as $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ (counting multiplicities), and an orthogonal set of eigenvectors $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ which form a basis for \mathbb{R}^n . Since the row sums of L are zero, the smallest eigenvalue λ_1 is always zero, and the corresponding eigenvector is $\mathbf{1} = (1, 1, \dots, 1)$. The multiplicity of the zero eigenvalue equals the number of connected components of the network. In particular, the second eigenvalue λ_2 is nonzero if and only if the network is connected, which is one of the most fundamental relations between the network structure and the spectrum of the connection operator.

For undirected networks, simple bounds can be given for the eigenvalues in

terms of the vertex degrees, which provide further insight into the relation between the structural and spectral properties. Let d_{\min} and d_{\max} denote, respectively, the smallest and the largest degree, and let λ_{\max} be the largest eigenvalue of the Laplacian. Then the following estimates are well-known (e.g. [17]):

$$\lambda_2 \leq \frac{n}{n-1}d_{\min} \leq \frac{n}{n-1}d_{\max} \leq \lambda_{\max} \leq 2d_{\max}. \quad (1)$$

Similarly, in terms of the average degree d_{avg} it can be shown that

$$d_{\text{avg}} < \lambda_{\max}; \quad (2)$$

see e.g. [18]. Note that the second eigenvalue does not have a simple bound from below in terms of the vertex degrees. This observation will be important later on, as we show that λ_2 can indeed be arbitrarily small among a class of networks having the same vertex degrees.

3 Spectral properties and synchronization

The nodes of a network are often dynamical systems evolving according to certain rules, and the links represent their pairwise interaction. A typical interaction type is diffusion, which forms the prototypical example where synchronization is observed [19], and naturally gives rise to the Laplacian operator L . It is thus no coincidence that the dynamical properties are closely related to the structural properties of the network. To focus on a well-known example, we consider the case of the so-called coupled map lattice [20]

$$x_i(t+1) = f(x_i(t)) + \sum_{j=1}^n a_{ij} [f(x_j(t)) - f(x_i(t))] \quad (3)$$

which we have written in a slightly more general form by allowing individual weights $a_{ij} \geq 0$ along the links instead of a common coupling strength for the whole network. Denoting $\mathbf{x} = (x_1, \dots, x_n)$ and $F(\mathbf{x}) = (f(x_1), \dots, f(x_n))$, the system (3) can be written in vector form

$$\mathbf{x}(t+1) = (I - L)F(\mathbf{x}(t)). \quad (4)$$

The network (3) is said to synchronize if $\lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| = 0$ for all i, j whenever the initial conditions belong to some appropriate open set¹. In this case, the system asymptotically approaches a synchronous state, where each node exhibits the same time evolution, $x_i(t) = s(t)$ for all i , or $\mathbf{x}(t) = \mathbf{1}s(t)$. It follows from (3) that $s(t+1) = f(s(t))$; i.e., the behavior of the nodes in the synchronous state is identical to their behavior in isolation². In this paper we

¹For chaotic synchronization there are some subtleties regarding the nature of the attraction and the open set of initial conditions; the interested reader is referred to [21] for a clarification of such issues. The details, however, will not be important for the derivation presented here.

²Here we neglect any coupling delays in the network. The synchronous solutions can be markedly different when delays are introduced; see [22, 23].

focus on chaotic synchronization, that is, the case when f has a compact chaotic attractor \mathcal{A} and s represents some dense (and necessarily unstable) orbit in \mathcal{A} . Assuming that f is continuously differentiable, small perturbations u about the solution $s(t)$ are governed by the equation $u(t+1) = f'(s(t))u(t)$, which has the solution

$$u(t) = u(0) \prod_{k=0}^{t-1} f'(s(k)).$$

Hence, the condition for local asymptotic stability of $s(t)$ is that

$$\lim_{t \rightarrow \infty} \prod_{k=0}^{t-1} |f'(s(k))| = 0. \quad (5)$$

While (5) would not hold for any solution s inside a chaotic attractor, it is always possible to find some sufficiently large number α such that

$$\lim_{t \rightarrow \infty} \prod_{k=0}^{t-1} e^{-\alpha} |f'(s(k))| = 0. \quad (6)$$

In fact, it is easy to see that (6) holds for all α satisfying

$$\alpha > \mu \triangleq \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} \log |f'(s(k))|; \quad (7)$$

where μ denotes the Lyapunov exponent.

Synchronization of coupled map lattices has been studied in more or less general forms; e.g., [7, 19, 24]. To find the corresponding conditions, one considers small perturbations $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{1}s(t)$, which are governed by the variational equation

$$\mathbf{u}(t+1) = f'(s(t))(I - L)\mathbf{u}(t).$$

Assuming that the eigenvectors of L form a basis for \mathbb{R}^n , the perturbations can be taken along an eigenvector of L , $\mathbf{u}(t) = p_i(t)\mathbf{u}_i$, where $i \geq 2$ since the perturbations along the direction $\mathbf{1}$ still yield a synchronous solution. The amplitude $p_i(t)$ along the i th eigenvector obeys

$$p_i(t+1) = f'(s(t))(1 - \lambda_i)p_i(t) = p_i(0) \prod_{k=0}^t f'(s(k))(1 - \lambda_i).$$

Thus, the system synchronizes if

$$\lim_{t \rightarrow \infty} \prod_{k=0}^{t-1} |f'(s(k))||1 - \lambda_i| = 0, \quad i = 2, \dots, n. \quad (8)$$

In view of (6) and (7), a sufficient condition for local synchronization is

$$\max\{|1 - \lambda_i| : i = 2, \dots, n\} < e^{-\mu}. \quad (9)$$

The significance of the simple condition (9) is twofold. Firstly, it separates the effects of the local (isolated) dynamics given by μ from the effects of the network structure given by the left-hand side. Therefore, an appropriate synchronizability measure for the network is

$$\sigma \triangleq \max\{|1 - \lambda_i| : i = 2, \dots, n\}, \quad (10)$$

smaller values of σ yielding synchronization for a larger class of functions f . Secondly, the role of the network structure on synchronizability is characterized by the spectrum of the Laplacian. The only assumption about L used above is the existence of n linearly independent eigenvectors, which is generically satisfied by matrices in $\mathbb{R}^{n \times n}$. Hence, σ can be used for comparing general networks with respect to their synchronizability, including directed and weighted ones, and even when they have different sizes.

For undirected weighted networks the synchronizability measure (10) simplifies to

$$\sigma = \max\{|1 - \lambda_2|, |1 - \lambda_{\max}|\}$$

where $\lambda_{\max} = \lambda_n$. Then three types of networks can be distinguished.

(a) All eigenvalues are less than or equal to 1. In this case synchronizability is determined solely by λ_2 , a larger value implying better synchronizability through the condition $\lambda_2 > 1 - e^{-\mu}$.

(b) All eigenvalues are larger than 1. In this case synchronizability is determined solely by λ_{\max} , a smaller value implying better synchronizability through the condition $\lambda_{\max} < 1 + e^{-\mu}$.

(c) $\lambda_2 \leq 1 \leq \lambda_{\max}$. In this case synchronizability depends on both λ_2 and λ_{\max} , higher values of λ_2 and smaller values of λ_{\max} implying better synchronizability through the condition

$$\frac{\lambda_2}{\lambda_{\max}} > \frac{1 - e^{-\mu}}{1 + e^{-\mu}}$$

Note that in a weighted network the eigenvalues contain information about the connection strengths. So, cases (a) and (b) can also be viewed as weakly and strongly coupled networks, respectively, whereas (c) can be thought of as the case of intermediate coupling strength. In this setting, the eigenratio λ_2/λ_{\max} has been used as a numerical measure of the synchronizability of networks. In all cases, the critical quantity here is often λ_2 since it can be arbitrarily small, whereas λ_{\max} can be bounded in terms of the largest vertex degree, as seen from (1).

4 Structural limitations to synchronization

The synchronizability of the network is directly related to its spectral properties by (9). Can the same be said about the statistical properties? We shall show that the answer is negative in general, although in certain cases some useful

information about synchronizability can be obtained. Throughout this section we deal with undirected networks.

Using (1), it is seen that

$$\frac{\lambda_2}{\lambda_{\max}} \leq \frac{d_{\min}}{d_{\max}}. \quad (11)$$

Therefore, a network whose smallest and largest degrees are very different is a bad synchronizer. For example, scale-free networks have poorer synchronizability in comparison to some other architectures, as observed in [25]. Note, however, that (11) does *not* imply that a more homogeneous degree distribution always means better synchronizability. (In fact, we later give examples where a higher degree homogeneity results in worse synchronizability.) For one thing, (11) is only an upper bound for the eigenratio. Moreover, the bound depends on the extreme degrees, whereas degree homogeneity (defined as the standard deviation of the degree distribution) is an average quantity, which may only loosely depend on the extreme degrees in large networks. In the following, we will use more sophisticated bounds on λ_2 and derive general structural limitations on synchronization. We will show that the effect of small structural changes on synchronization need *not* average out within the large network structure, and therefore may not be captured by the average network properties.

For notation, let V denote the set of vertices of an undirected network G , and let $S \subset V$ be a subset of vertices, with $V - S$ denoting its complement, and $|S|$ its cardinality. Define

$$|\partial S| = \sum_{i \in S} \sum_{j \in V-S} a_{ij}.$$

In words, $|\partial S|$ is the (weighted) number of edges between S and its complement. The isoperimetric number $i(G)$ of a graph G is defined by

$$i(G) = \min \left\{ \frac{|\partial S|}{|S|} : S \subset V, 0 < |S| \leq \frac{n}{2} \right\}. \quad (12)$$

The computation of $i(G)$ is an NP-hard problem [26]. However, an important result in graph theory gives a lower bound for the isoperimetric number in terms of the second eigenvalue of the Laplacian, namely, $i(G) \geq \frac{1}{2}\lambda_2$. We turn the table around, and use this result and (12) to estimate λ_2 as

$$\lambda_2 \leq 2 \frac{|\partial S|}{|S|} \quad (13)$$

where S is any subset of vertices satisfying $0 < |S| \leq n/2$.

The estimate (13) holds the key to understanding why the statistical properties of the network can fail to determine λ_2 . The important observation is that the bound on λ_2 is determined by the properties of some subgraph S and not in general by the graph itself. In particular, S can be very small compared to the whole graph, in which case the statistical properties of the graph need not be reflected in S , although the latter plays a crucial role in constraining

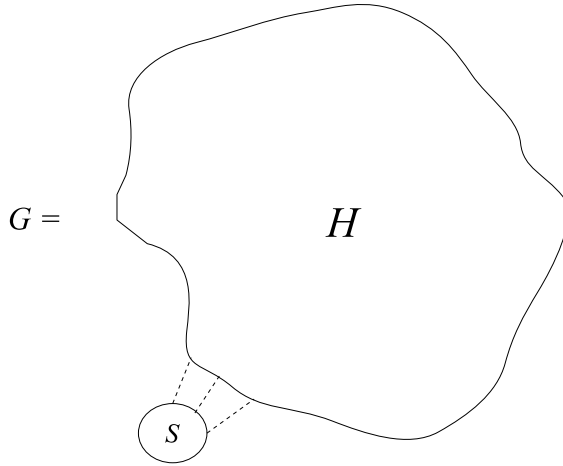


Figure 1: The statistical properties of the graph G is determined by the huge part H , while the value of λ_2 is independently constrained by the small subgraph S .

the value of λ_2 . Figure 1 illustrates the idea in intuitive terms. Suppose in the graph G we identify a huge part H and a much smaller part S . (Alternatively, we can imagine the possibility of appending a small set of nodes S to an existing graph, which is a realistic scenario if one considers time-varying connections which might come on and off [21, 27, 28]). By (13), the value of λ_2 is constrained by the properties of S . However, all the gross statistical properties of G are determined by H . If H is any graph which is claimed to have good synchronizability, we can force G to have poor synchronizability by appending S to H . In other words, for large networks, the synchronizability of G and H can be very different, although many of their statistical properties are essentially the same. For instance, if S consists of 20 nodes and is connected to H by one link, then by (13) $\lambda_2 \leq 0.1$ regardless of how H is chosen. Furthermore, λ_2/λ_{\max} can be considerably smaller, especially if the average degree is high (viz. (2)), which shows that a large average degree can actually impede synchronizability. This example also illustrates the phenomenon observed in [18]; namely, when two networks are combined by adding some links between them, the synchronizability of the overall network decreases as the synchronizability of individual networks is increased.

We have established that it is the local structures, described by the sets S , that constrain the synchronizability, regardless of the global properties of the network. Such local structures are conspicuous in certain types of networks while they may not be so obvious in others. For example, the situation shown in Figure 1 is typical for traffic or transportation networks, where traffic is much denser within cities than between them, and for interacting brain areas, where intracortical connectivity is higher than inter-areal connections. We note, however, that such local structural constraints need not exist in every network.

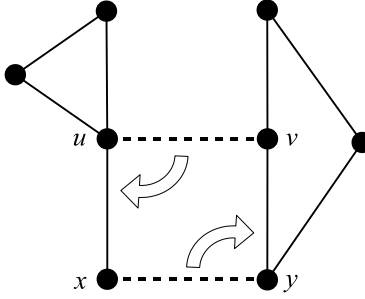


Figure 2: Link switching: The links uv and xy are deleted, and new links ux and vy are added. The vertex degrees remain unchanged after the operation.

For instance, if the minimum degree is much larger than $n/2$, then the ratio $|\partial S|/|S|$ will be large for any subset S satisfying $|S| \leq n/2$. However, such networks are very densely connected (the total number of links being at least $n^2/4$, which is about one half of that of a complete graph), whereas most real-world networks are much sparser. Hence, if one considers networks which are not too densely connected, it turns out that within essentially any family of graphs having the same degree distribution, there exist graphs containing subsets S for which $|\partial S|/|S|$ is small, and so the graph has a small second eigenvalue λ_2 . For a detailed mathematical proof the reader is referred to [29].

Without going into technical details, we here illustrate the essential ideas to show that networks with the same degree distribution can have very different synchronizability. A useful notion for this purpose is the use of link switching to vary network properties without altering the vertex degrees [30]. As depicted in Figure 2, link switching refers to the operation where, given two pairs of neighboring nodes u, v and x, y , one breaks the links uv and xy and replaces them by the links ux and vy . The operation leaves the vertex degrees unchanged. As in the particular case of Figure 2, the resulting network can be disconnected, i.e. λ_2 becomes zero after the switch. Another possibility is to link u to y and v to x , which keeps the network connected. It follows that λ_2 can be changed by link switching, which makes it clear that the degree distribution does not determine λ_2 .

As an example, we consider a circularly arranged set of nodes where each node is connected to its k nearest neighbors on each side (Figure 3). Such circular structures have been heavily used in numerical studies of the small-world effect, instigated by [31]. We start with 200 nodes, where each node is connected to its 5 nearest neighbors on each side, and randomly switch pairs of links so that the degree of each node remains the same³. It is seen from Figure 4 that after only a few switches the eigenratio increases by more than a

³The construction is similar to that in [15], which adds random links to a circular arrangement of nodes, whereas here we use link switching to keep the vertex degrees unchanged.

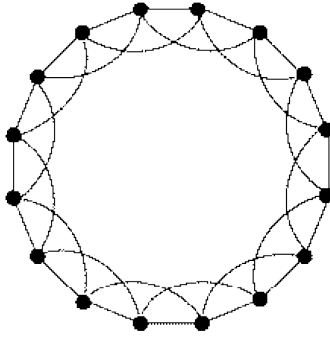


Figure 3: Circular arrangement of nodes.

factor of 10. In other words, the circularly arranged network has very different synchronizability characteristics than a typical regular network⁴, and leads to an underestimation of the synchronizability for the latter. In fact, randomly constructed large regular networks are typically expanders (see Section 5), i.e., their eigenvalues λ_2 can be bounded from below by a positive number.

We next give a concrete construction for obtaining very good and very bad synchronizing networks having identical vertex degrees. Consider regular networks of $n = 2m$ nodes where each node has degree $m - 1$ ⁵. We separate the nodes into two groups of m elements, and distribute the $m(m - 1)$ links in two different ways, as shown in Figure 5. In the first graph G_1 , all the links are across the two groups, and there are no connections within a group. In mathematical terms, its adjacency matrix is given by

$$A_1 = \begin{bmatrix} 0 & J_m - I_m \\ J_m - I_m & 0 \end{bmatrix},$$

where J_m denotes the $m \times m$ matrix whose every element is 1, and I_m is the $m \times m$ identity matrix. In the second graph G_2 , we start by putting all the links within a group, ending up with a disconnected network with two components (the two pentagonal shapes in Figure 5). The corresponding adjacency matrix is

$$A_2 = \begin{bmatrix} J_m - I_m & 0 \\ 0 & J_m - I_m \end{bmatrix}.$$

To obtain a connected network, we use link switching to replace one link within each component (shown by dotted lines) with a link connecting the two components. In this way, for each m we construct two different graphs with the

⁴A regular network is one where each vertex has the same degree.

⁵The degree k of each node can also be smaller than $m - 1$ without changing the subsequent argument. However, for $k > m - 1$ the construction for G_2 fails. Nevertheless, an average degree of $m - 1$ for $2m$ nodes already implies a well-connected network, the number of links $m(m - 1)$ being about one half of that of a complete graph, $m(2m - 1)$. Since real networks are usually much sparser than that, it suffices to consider $k \leq m - 1$.

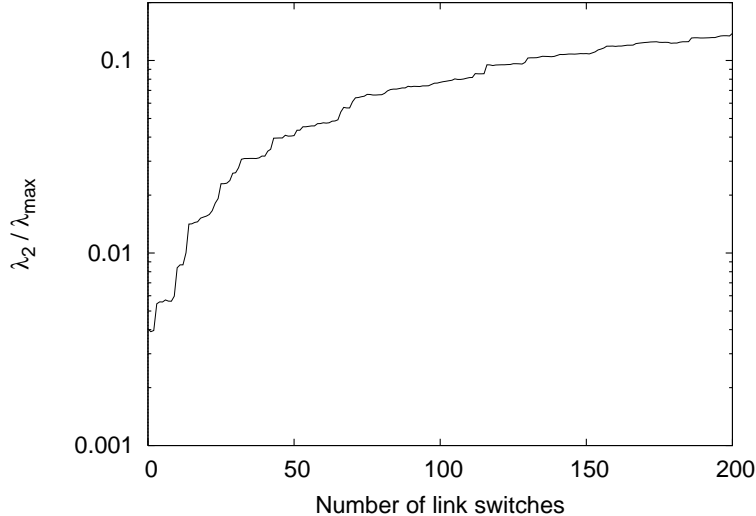


Figure 4: Improvement of synchronizability by random link switches, starting from a circularly arranged regular network of 200 nodes with vertex degree equal to 10.

same degree distribution, and having the maximum homogeneity of vertex degrees, since each one is a regular network. However, these two networks have completely different synchronizability characteristics. Indeed, the first network G_1 is related to the so-called complete bipartite graph. (If each vertex degree were m we would have exactly a complete bipartite graph, in which case $\lambda_2 = m = \lambda_{\max}/2$.) The eigenvalues for G_1 are $\lambda_2 = m - 2$ and $\lambda_{\max} = 2(m - 1)$; so the ratio λ_2/λ_{\max} increases and tends to $1/2$ as m gets large. For the second network G_2 , we use (13) to estimate $\lambda_2/\lambda_{\max} \leq \lambda_2 \leq 4/m$, which tends to zero as m gets large. Figure 6 shows the ratio λ_2/λ_{\max} for the two networks. It can be seen that a whole range $(0, 0.5)$ of values for λ_2/λ_{\max} can be generated using only regular graphs, which include very good as well as very bad synchronizers. Furthermore, since all these graphs have maximally homogeneous degree distribution, it is clear that the homogeneity of the degree distribution does not determine synchronizability.

The argument above also shows that the average degree fails to determine synchronizability: The average degree $m - 1$ of both networks increases with m ; however, this increase results in a better synchronizability for G_1 and a worse synchronizability for G_2 . For a similar example which uses non-regular networks, see [18].

In closing this section, we mention that there are alternative definitions of the Laplacian, given by $I - D^{-1/2}AD^{-1/2}$ or $I - D^{-1}A$. The foregoing arguments apply also to the second eigenvalue of these matrices, with the same conclusions; see [29]. Also for directed networks, one can use the idea of Fig. 1 to show that appending a small set of vertices S to an existing graph disrupts synchronizability

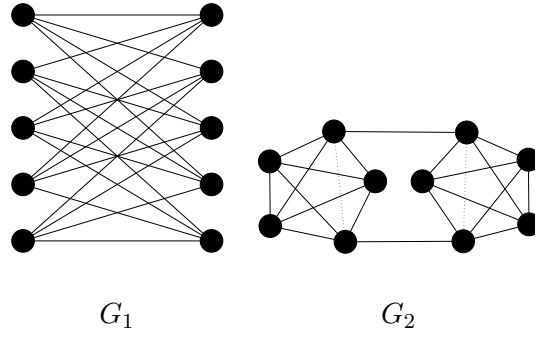


Figure 5: Two graphs with the same degree distribution and maximal degree homogeneity, but very different synchronizability.

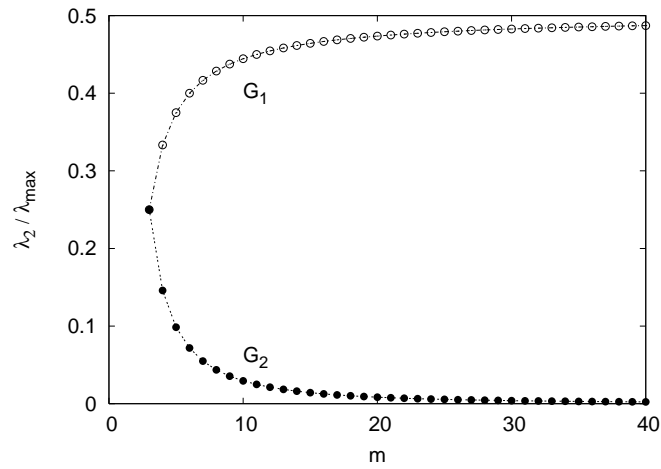


Figure 6: The eigenratio for the two graphs of Figure 5.

without affecting average statistical properties of the graph too much. Indeed, it is always possible to choose an S having as few as two vertices and containing no directed spanning tree. Since the resulting graph G also contains no directed spanning tree, it is incapable of chaotic synchronization [32].

5 Discussion and conclusion

The second eigenvalue λ_2 of the Laplacian is an important invariant for undirected graphs. Also called the *algebraic connectivity* or the *spectral gap*, it has a special place within the Laplacian spectrum, and is deeply related to many structural graph properties. For example, it comes up in random walks on graphs, and consequently in epidemic spreading, as well as robustness against edge and vertex removal (cut problems). Hence, the result that λ_2 is not controlled by statistical properties of the network has significance that goes beyond synchronization.

It is known from graph theory that there do not exist generally useful lower bounds for the eigenvalue λ_2 . Some estimates can be obtained asymptotically and in a probabilistic sense, i.e., almost surely as the network size goes to infinity, and have been applied to study the asymptotic behavior of synchronizability in power-law networks [33]. A nice mathematical result derived in [34] for undirected random networks with given expected degrees states that

$$\max_{i \geq 2} |1 - \lambda_i| \leq (1 + o(1)) \frac{4}{\sqrt{w_{\text{avg}}}} + \frac{g(n) \log^2 n}{w_{\text{min}}} \quad (14)$$

where w_{avg} and w_{min} are the expected values of the average and minimum degrees, respectively, n is the network size, and $g(n)$ is some slow-growing function of n . Note that the left-hand side of (14) is the precisely the network synchronizability measure σ defined in (10). The implication is that, under conditions that the second term on the right is negligible compared to the first, one has essentially

$$\sigma \leq \frac{4}{\sqrt{w_{\text{avg}}}} \quad \text{and} \quad \frac{\lambda_2}{\lambda_{\text{max}}} \geq \frac{1 - 4/\sqrt{w_{\text{avg}}}}{1 + 4/\sqrt{w_{\text{avg}}}}. \quad (15)$$

as $n \rightarrow \infty$. These estimates in turn would imply that in the asymptotic limit a typical network with a large average degree (namely, $\sqrt{w_{\text{avg}}}$ at least as large as $4e^\mu$ by (9)) is a good synchronizer. However, some care is needed in using (14) to derive conclusions about finite graphs. Since the $o(1)$ term has no bounds in terms of graph size, it need not be small for a large but finite graph. Moreover, to neglect the second term on the right-hand side of (14), it is necessary that the expected minimum degree w_{min} grow faster than $\log^2 n$ as $n \rightarrow \infty$. In other words, (15) can be justified only for graphs for which both the size and the minimum degree are very large. Unfortunately, real networks of interest are both finite and sparse, and as our results indicate, (15) does not necessarily hold for these networks. In fact, (15) is false even when we restrict ourselves to a smaller class of graphs by imposing additional restrictions in terms of network statistics,

such as fixing the degree distribution or requiring high degree homogeneity. Any such class would still contain a bad synchronizer with positive probability. This is the precise meaning of our statement that statistical properties do not suffice to determine synchronizability.

Clearly, the true criteria for classifying networks with respect to synchronizability involve the Laplacian spectrum, using some measure such as (9)⁶. A closely related notion in graph theory is that of expander graphs. Informally, these are families of sparse graphs with high connectivity, so that the isoperimetric number defined by (12) is bounded from below by some positive number. In this context, the isoperimetric number $i(G)$ is also called the *expansion constant*. Recall that our arguments for identifying poorly-synchronizing networks are based on showing that $i(G)$ can be small. Hence, in terms of the expansion properties of networks, our results imply that the gross statistical properties of networks do not suffice to characterize expander families.

There are many factors that contribute to the difficulties of studying complex networks. The number of different networks of size n increases dramatically with n , which already makes it hard to obtain the relations between the numerous network properties based on numerical simulations alone. One might contend that numerical simulations give information about “typical” networks in a certain class, but the mathematical proof of such assertions remains an open problem. Moreover, the probability distributions from which networks with specific statistical properties are drawn have rarely been specified in the literature. On the other hand, a more subtle question worth consideration is whether “typical” networks or properties carry all the information that one should be interested in. The question is more meaningful in the context of the complex networks found in nature, such as the human brain or metabolic networks, which have very distinct functions and have evolved after a long period of time into their present state. It is a certainly intriguing possibility that their function may be related to, say, their degree distribution. However, it is hardly warranted to claim that the function is *only* a consequence of that particular distribution. Such a claim would imply that all networks with the same degree distribution are similar at the functional level, which downplays the role of the millions of years of evolution behind natural networks. In fact, one could argue that many natural networks may be necessarily “atypical” in certain sense, if evolution points along the direction of some optimization process. Hence, when considering networks with such unique functions, the relevant features may be those that make the network distinguished rather than typical, within the considered class of networks.

In conclusion, dynamical processes on networks, and in particular synchronization, are intimately related to the eigenvalues of the coupling operator. As shown in the present paper, the gross statistical properties of networks do not generally suffice to determine the spectrum. In other words, the eigenvalues are among the intrinsic network features which determine the dynamics and which are not derivable from the statistical characteristics. Consequently, the spectral

⁶It should however be kept in mind that (9) is a sufficient but not a necessary condition for synchronization of chaotic systems.

network properties deserve more attention in the basic description and study of complex networks.

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